Teorem: A $n\times n$ tipinde olsun. A'nın özdeğerleri karakteristik polinomunun kökleridir.

İspat: λ A nın bir özdegeri ve x, λ ya karşılık gelen bir özvektörü olsun.

$$A x = \lambda . x \iff A x = (\lambda I_n) . x$$
$$\iff \underline{(\lambda I_n - A) x = \vec{0}}$$

($x \neq \vec{0}$ istiyoruz.) $n\times n$ tipinde bir homojen lineer denklem sisteminin aşikâr olmayan çözümleri olması için gerek ve yeter şart katsayılar matrisi $(\lambda I - A)$ nın determinantının sıfır olmasıdır.

$$p_A(\lambda) = \det(\lambda I - A) = 0 \qquad \square$$

Örnek: $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

∓ 1
∓ 2
∓ 3
∓ 6

matrisinin karakteristik polinomu

$$p(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$
$$= (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

A nın özdeğerleri $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

Özvektörlerin Bulunuşu:

$\lambda_1 = 1$ e karşılık gelen:

$$Ax = \lambda x = (\lambda I)x \Leftrightarrow \underbrace{(\overset{\lambda_1}{\lambda} I - A)x}_{=0}$$

$$\begin{bmatrix} 1-1 & -2 & +1 \\ -1 & 1 & -1 \\ -4 & 4 & 1-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ -4 & 4 & -4 & 0 \end{array}\right]$$

$$\underset{-S_2 \leftrightarrow S_1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -4 & 4 & -4 & 0 \end{array}\right]$$

$$A \cdot v_1 = 1 \cdot v_1$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}}_{v_1} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}}_{1 \cdot v_1}$$

sağlama.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -4 & 4 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_2 = \frac{x_3}{2} = \frac{s}{2}$$

$$x_1 = x_2 - x_3 = \frac{s}{2} - s = \frac{-s}{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s/2 \\ s/2 \\ s \end{bmatrix} = \left(-\frac{s}{2}\right) \cdot \underbrace{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}}_{v_1}$$

v_1, $\lambda_1 = 1$ özdeğerine karşılık gelen bir özvektördür.

$\lambda_2 = 2$ için

$$(\lambda_2 I - A) = \begin{bmatrix} 2-1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & 2-5 \end{bmatrix} \qquad (\lambda_2 I - A)x = 0$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ -4 & 4 & -3 & 0 \end{array}\right] \begin{array}{c} S_1+S_2 \\ 4S_1+S_3 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 \end{array}\right]$$

$$x_2 = \frac{x_3}{4} = \frac{s}{4}, \quad x_3 = s$$

$$x_1 = 2x_2 - x_3 = \frac{s}{2} - s = -\frac{s}{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s/2 \\ s/4 \\ s \end{bmatrix} = \frac{s}{4}\underbrace{\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}}_{v_2}$$

$$A \cdot v_2 = 2 \cdot v_2$$

$\lambda_3 = 3$ için

$$(\lambda_3 I - A)x = 0 \iff \left[\begin{array}{ccc|c} 3-1 & -2 & 1 & 0 \\ -1 & 3 & -1 & 0 \\ -4 & 4 & 3-5 & 0 \end{array}\right]$$

$$\begin{array}{c} -S_2 \leftrightarrow S_1 \\ \sim \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ -4 & 4 & -2 & 0 \end{array}\right] \begin{array}{c} -2S_1+S_2 \\ \sim \\ 4S_1+S_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & -8 & 2 & 0 \end{array}\right]$$

$$\begin{array}{c} 1/4\, S_2 \to S_2 \\ 2S_2+S_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = \dfrac{x_3}{4}$, $x_3 = s$ için

$x_2 = \dfrac{s}{4}$

$x_1 = 3x_2 - x_3 = 3\dfrac{s}{4} - s = \dfrac{-s}{4}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s/4 \\ s/4 \\ s \end{bmatrix} = \left(\frac{s}{4}\right) \underbrace{\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}}_{\to v_3}$$

$A \cdot v_3 = 3 \cdot v_3$

Örnek: $L: P_2 \to P_2$, $L(at^2+bt+c) = -bt-2c$

lineer operatörünün özdeğer ve özvektörlerini

P_2 nin $S=\{t-1, 1, t^2\}$ sıralı bazına göre temsilci

matrisinden yararlanarak bulunuz.

L nin S e göre temsilcisi için:

$$A = \left[[L(t-1)]_S \,\vdots\, [L(1)]_S \,\vdots\, [L(t^2)]_S \right]$$

$$A = \left[[-t+2]_S \,\vdots\, [-2]_S \,\vdots\, [0] \right] = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 & 0 \\ -1 & \lambda+2 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = (\lambda+1)(\lambda+2)\lambda = 0$$

A nın özdeğerleri: $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = 0$

Özvektörler:

$\lambda_1 = -1$ için: $(\lambda_1 I - A)\,x = 0 \iff \left[\begin{array}{ccc|c} -1+1 & 0 & 0 & 0 \\ -1 & -1+2 & 0 & 0 \\ 0 & 0 & \underbrace{-1} & 0 \end{array}\right]$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A$$

$x_3 = 0\,, \quad x_1 = x_2 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = v_1$$

$A \cdot v_1 = (-1) \cdot v_1$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = (-1) \cdot v_1$$

$L : P_2 \to P_2\,; \quad A : \mathbb{R}^3 \to \mathbb{R}^3$

$[p_1]_S = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = v_1$

$p_1(t) = \quad \leftarrow$

$1 \cdot (t-1) + 1 \cdot 1 = t$

$p_1(t) = t$

$\lambda_1 = -1$ e karşılık gelen özvektör.

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A \Rightarrow (\lambda_2 I - A)x = 0$$

$\lambda_2 = -2$ için:

$$\left[\begin{array}{ccc|c} -2+1 & 0 & 0 & 0 \\ -1 & -2+2 & 0 & 0 \\ 0 & 0 & \underset{\lambda_2}{-2} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{c|c|c|c} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

koşul yok

$x_1 = 0$

$x_3 = 0$

$x_2 = s =$ serbest

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix}$$

$$s \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_2}$$

$v_2 = [p_2]_s = \underline{1}$, $\lambda_2 = -2$ ye karşılık özvektörü.

$\lambda_3 = 0$ için: $A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0+1 & 0 & 0 & 0 \\ 1 & 0+2 & 0 & 0 \\ 0 & 0 & \underset{\text{koşul yok}}{\underbrace{0}} & 0 \end{array}\right]$$

$x_1 = 0$

$x_2 = 0$

$x_3 = \text{serbest} = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{v_3}$$

$$v_3 = \left[p_3 \right]_S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow p_3 = 0.(t-1) + 0.1 + 1.t^2$$

$p_3 = t^2$ $\lambda_3 = 0$ özd. karşılık gelen öz vektördür.

$$A \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot v_3$$

Örnek: $L: P_2 \to P_2$, $L(at^2+bt+c) = c - at^2$

lineer operatörünün özdeğer ve özvektörlerini, $S = \{t^2+1, t, 1\}$ sıralı bazına göre temsilcisinden yararlanarak bulunuz.

$$A = \left[[L(t^2+1)]_S \,\vdots\, [L(t)]_S \,\vdots\, [L(1)]_S \right] = \left[[1-t^2]_S \,\Big|\, [0]_S \,\Big|\, [1]_S \right]$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}; \quad |\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda & 0 \\ -2 & 0 & \lambda-1 \end{vmatrix} = (\lambda)(\lambda-1)(\lambda+1)$$

Özdeğerler: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$

Özvektörler: $\lambda_1 = 0$ için:

$$(\lambda_1 I - A)x = 0 \iff \left[\begin{array}{ccc|c} \underline{0+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0-1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_3 = 0 \\ x_2 = \text{serbest} = s \end{array}$$

$$\underset{\downarrow}{0}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_1}$$

$$\boxed{v_1 = \begin{bmatrix} p_1 \end{bmatrix}_s, \; p_1 = t}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$\lambda_2 = 1$ için: $(\lambda_2 I - A)x = 0 \iff \left[\begin{array}{ccc|c} 1+1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1-1 & 0 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & \underset{\downarrow}{0} & 0 \end{array}\right] \begin{array}{l} \rightarrow x_1 = 0 \\ \rightarrow x_2 = 0 \\ x_3 = \text{serbest} = s \end{array} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$

$[p_2]_s = v_2 \Rightarrow p_2 = 1 \cdot 1 = 1$, $\lambda_2 = 1$ için özvekt.

$\lambda_3=-1$ için: $(\lambda_1 I - A)x = 0 \Leftrightarrow \left[\begin{array}{ccc|c} -1+1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & (-1-1) & 0 \end{array}\right]$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} = A$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \quad x_2 = 0, \quad x_1 = -x_3 = -s, \quad (x_3 = s \text{ için})$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{v_3}$$

$$v_3 = [p_3]_S \Rightarrow p_3 = (-1)\cdot(t^2+1) + 1\cdot 1 = \underline{\underline{-t^2}}$$

$\lambda_3 = -1$ e karşılık gelen özvektör

Örnek: $A=\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ matrisinin karakteristik polinomunu, özdeğer ve karşılık gelen özvektörlerini bulunuz.

Karakt. Pol: $|\lambda I - A|$

$$= \begin{vmatrix} \lambda-2 & -2 & -3 \\ -1 & \lambda-2 & -1 \\ -2 & 2 & \lambda-1 \end{vmatrix} \underset{(-2).s_2+s_1}{\overset{(\lambda-2).s_2+s_3}{=}} \begin{vmatrix} 0 & \left[(\lambda-2)^2-2\right] & (-3+2-\lambda) \\ -1 & \lambda-2 & -1 \\ 0 & 6-2\lambda & \lambda+1 \end{vmatrix}$$

$$= \underbrace{(-1).(-1)^{2+1}}_{\text{1. sütun b. kofaktör açılımı}} . \begin{vmatrix} \lambda^2-4\lambda+2 & -1-\lambda \\ 6-2\lambda & \lambda+1 \end{vmatrix}$$

$$= \lambda^3-4\lambda^2+2\lambda+\lambda^2-4\lambda+2+(6-2\lambda)(\lambda+1)$$

$$= \lambda^3-3\lambda^2-2\lambda+2+(-2\lambda^2)-2\lambda+6\lambda+6$$

$$= \lambda^3-5\lambda^2+2\lambda+8 = P_A(\lambda)$$ Karakt. polinom.

$$\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$$

$$\downarrow$$

$$(+2)^3 - 5.4 + 2.2 + 8 = 20 - 20 \; ; \Rightarrow \quad \lambda = 2 \text{ bir kök.}$$

$$\lambda^3 - 5\lambda^2 + 2\lambda + 8 = (\lambda - 2) \cdot q(\lambda)$$

$$\lambda^3 - 5\lambda^2 + 2\lambda + 8 = P_A(\lambda)$$

$$\begin{array}{r|l} \lambda^3 - 5\lambda^2 + 2\lambda + 8 & \lambda - 2 \\ \hline - \quad \lambda^3 - 2\lambda^2 & \lambda^2 - 3\lambda - 4 \\ \hline -3\lambda^2 + 2\lambda + 8 & \\ - \quad -3\lambda^2 + 6\lambda & \\ \hline -4\lambda + 8 & \\ -4\lambda + 8 & \\ \hline 0 & \end{array}$$

$$\underbrace{\begin{aligned} P_A(\lambda) &= (\lambda - 2).(\lambda^2 - 3\lambda - 4) \\ &= (\lambda - 2)(\lambda - 4)(\lambda + 1) \end{aligned}}_{\text{karakt. polinomun çarpanları;}}$$

$\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = -1$ A matrisinin özdeğerleridir

Özvektörlerin Bulunuşu:

$$\lambda_1 = 2 \Rightarrow (\lambda_1 I - A)X = 0 \Leftrightarrow \left[\begin{array}{ccc|c} 2-2 & -2 & -3 & 0 \\ -1 & 2-2 & -1 & 0 \\ -2 & 2 & (2-1) & 0 \end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 0 & -2 & -3 & 0 \\ -1 & 0 & -1 & 0 \\ -2 & 2 & 1 & 0 \end{array}\right] \overset{-S_2 \Leftrightarrow S_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 0 \\ -2 & 2 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad x_2 = -\frac{3}{2} \cdot x_3, \; x_3 = s, \quad x_1 = -s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \cdot \begin{bmatrix} -1 \\ -3/2 \\ 1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} -1 \\ -3/2 \\ 1 \end{bmatrix},$$

$\lambda_1 = 2$ özdeğerine karşılık gelen bir özvektördür.

$\lambda_2 = 4$ için: $(\lambda I - A)x = 0 \Leftrightarrow$

$$\left[\begin{array}{ccc|c} 4-2 & -2 & -3 & 0 \\ -1 & 4-2 & -1 & 0 \\ -2 & 2 & 4-1 & 0 \end{array}\right] \underset{-S_2 \leftrightarrow S_1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -2 & -3 & 0 \\ -2 & 2 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -5 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right]$$

$x_2 = \frac{5}{2} \cdot x_3$, $x_3 = s$ için

$x_2 = \frac{5}{2} s$

$x_1 = 2 \cdot x_2 - x_3 = 4s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 4 \\ 5/2 \\ 1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 4 \\ 5/2 \\ 1 \end{bmatrix},$$

$\lambda_2 = 4$'e karşılık gelen bir özvektördür.

$\lambda_3 = -1$ için: $(\lambda I - A)x = 0 \Leftrightarrow$

$$\left[\begin{array}{ccc|c} -1-2 & -2 & -3 & 0 \\ -1 & -1-2 & -1 & 0 \\ -2 & 2 & -1-1 & 0 \end{array}\right] \underset{\sim}{-S_2 \leftrightarrow S_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -3 & -2 & -3 & 0 \\ -2 & 2 & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} 3S_1+S_2 \\ 2S_1+S_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_2 = 0 \\ x_1 = -x_3 = -s \quad (x_3 = s \text{ için}) \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

$\lambda_3 = -1$ 'e karşılık gelen bir özvektördür.